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Numerical Solutions of Linear Volterra and Fredholm Integral Equations of Second kind using Adomian Decomposition Method and Variational Iteration Method

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Abstract

This research addresses the development of more advanced and efficient methods for integral equations and integro-differential equations such as successive approximations method, Laplace transform method, spline collocation method, Runge-Kutta method, and others have been used to handle Volterra integral equations. In our work, we apply the Adomian decomposition method (ADM) and the variational iteration method to handle Volterra integral equations of second kind and Fredholm equations of second kind and the Adomian decomposition method (ADM) and the variational iteration method to handle Volterra integral equations of second kind and Fredholm equations of second kind are applied.

Keywords: Numerical Solutions, Linear Volterra, Fredholm Integral Equations, Adomian Decomposition, Variational Iteration.

Introduction

Since many physical problems are modeled by integro-differential equations, the numerical solutions of such integro-differential equations have been highly studied by many authors.

In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integral equations and integro-differential equations such as successive approximations method, Laplace transform method, spline collocation method, Runge-Kutta method, and others have been used to handle Volterra integral equations. In our work, we apply the Adomian decomposition method(ADM) and the variational iteration method to handle Volterra integral equations of second kind and Fredholm equations of second kind.

An integral equation is the equation in which the unknown function $u(x)$ appears inside an integral sign [1–5]. The most standard type of integral equation in $u(x)$ is of the form

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt \qquad (1.1)$$

Where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $K(x,t)$ is a known function, of two variables x and t, called the kernel or the nucleus of the integral equation. The unknown function $u(x)$ that will be determined appears inside the integral sign. In many other cases, the unknown function $u(x)$ appears inside and outside the integral sign.

The functions $f(x)$ and $K(x,t)$ are given in advance. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants, or mixed. Integral equations appear in many forms. Two distinct ways that depend on the limits of integration are used to characterize integral equations, namely:

a. If the limits of integration are fixed, the integral equation is called a Fredholm integral equation given in the form: $u(x) = f(x) + \lambda \int_{a}^{b} K(x,t)u(t)dt \qquad (1.2)$

Where a and b are constants.

b. If at least one limit is a variable, the equation is called a Volterra integral Equation given in the form: $u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt$ (1.3)

Moreover, two other distinct kinds, that depend on the appearance of the unknown function $u(x)$, are defined as follows:

i. If the unknown function $u(x)$ appears only under the integral sign of Fredholm or Volterra equation, the integral equation is called a First kind Fredholm or Volterra integral equation respectively.

ii. If the unknown function(x) appears both inside and outside the integral sign of Fredholm or Volterra equation, the integral equation is called a Second kind Fredholm or Volterra equation integral equation respectively. In all Fredholm or Volterra integral equations presented above, if $f(x)$ is identically zero, the resulting equation:

$$u(x) = \lambda \int_a^b K(x,t)u(t)dt \quad or \quad u(x) = \lambda \int_a^x K(x,t)u(t)dt \tag{1.4}$$

Is called Homogeneous Fredholm or HomogeneousVolterra integral equation respectively. It is interesting to point out that any equation that includes both integrals and derivatives of the unknown function $u(x)$ is called Integro-differential equation. The Fredholm integro-differential equation is of the form:

$$u^k(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \qquad u^k = \frac{d^k u}{dx^k} \tag{1.5}$$

However, the Volterra integro-differential equation is of the form:

$$u^k(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt, \qquad u^k = \frac{d^k u}{dx^k} \tag{1.6}$$

Classification of Integral Equations

Integral equations appear in many types. The types depend mainly on the limits of integration and the kernel of the equation. In this text, we will be concerned on the following types of integral equations.

Volterra Integral Equations

In Volterra integral equations, at least one of the limits of integration is a variable. For the first kind Volterra integral equations, the unknown function $u(x)$ appears only inside integral sign in the form:

$$f(x) = \int_a^x K(x,t)u(t)dt \tag{1.7}$$

However, Volterra integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt \tag{1.8}$$

Examples of the Volterra integral equations of the first kind are

$$a)\ xe^x = \int_0^x e^{t-x}u(t)dt$$

$$b)\ 5x^2 + x^3 = \int_0^x (5 + 3x - 3t)dt$$

However, examples of the Volterra integral equations of the second kind are

$$a)\ u(x) = 1 - \int_0^x u(t)dt$$

$$b)\ u(x)x + \int_0^x (x-t)u(t)dt$$

Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x)$may appear only inside integral equation in the form:

$$f(x) = \int_a^b K(x,t)u(t)dt \qquad (1.9)$$

This is called Fredholm integral equation of the first kind. However, for Fredholm integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (1.20)$$

Examples of the two kinds are given by

$a)\ \frac{sinx - xcosx}{x^2} = \int_0^1 \sin(xt)\, u(t)dt$

$b) u(x) = x + \frac{1}{2}\int_{-1}^{1}(x-t)u(t)dt$

Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations [6, 7] arise from parabolic boundary value problems, from the mathematical modeling of the spatio-temporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)dt + \lambda_2 \int_a^x K_2(x,t)u(t)dt \qquad (1.21)$$

And

$$u(x) = f(x,t) + \lambda \int_0^t \int F(x,t,\xi,\tau,u(\xi,\tau)\, d\xi d\tau, \quad (x,t) \in \Omega \times [0,T] \qquad (1.22)$$

Where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau)$ are analytic functions on $D = \Omega \times [0,T]$ and Ω is a closed subset of $\mathbb{R}^n, n = 1,2,3$. It is interesting to note that (1.21) contains disjoint Volterra and Fredholm integral equations, whereas (1.22) contains mixed Volterra and Fredholm integral equations. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind

Examples of the above two types are given by:

$$a)\ u(x) = 6x + 3x^2 + 2 - \int_0^x xu(t)dt - \int_0^1 tu(t)dt$$

$$b)\ u(x,t) = x + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)d\xi d\tau$$

Singular Integral Equations

Volterra integral equations of the first kind [4, 7]

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt \qquad (1.23)$$

Or of the second kind

$$u(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t)u(t)dt \qquad (1.24)$$

Are called singular if one of the limits of integration $u(x), h(x)$ or both are infinite. Moreover, the previous two equations are called singular if the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration. In this text, we will focus our concern on equations of the form:

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \qquad 0 < \alpha < 1, \qquad (1.25)$$

Or of the second kind: $u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \qquad 0 < \alpha < 1,$ (1.26)

The last two standard forms are called generalized Abel's integral equation and weakly singular integral equations respectively. For $\alpha = \frac{1}{2}$, the equation:

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt \qquad (1.27)$$

is called the Abel's singular integral equation. It is to be noted that the kernel in each equation becomes infinity at the upper limit $t = x$.

Examples of Abel's integral equation, generalized Abel's integral equation, and the weakly singular integral equation are given by

$$a)\ \sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt,$$

$$b)\ x^3 = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt,$$

$$c)\ u(x) = 1 + \sqrt{x \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt,}$$

Classification of Integro-Differential Equations

Integro-differential equations appear in many scientific applications, especially when we convert initial value problems or boundary value problems to integral equations. The integro-differential equations contain both integral and differential operators. The derivatives of the unknown functions may appear to any order. In classifying integro-differential equations, we will follow the same category used before.

Fredholm Integro-Differential Equations

Fredholm integro-differential equations appear when we convert differential equations to integral equations. The Fredholm integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x),\ \ n \geq 1$ inside and outside the integral sign respectively. The limits of integration in this case are fixed as in the Fredholm integral equations. The equation is labeled as integro-differential because it contains differential and integral operators in the same equation. It is important to note that initial conditions should be given for Fredholm integro-differential equations to obtain the particular solutions. The Fredholm integro-differential equation appears in the form:

$$u^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (1.28)$$

Where $u^{(n)}$ indicates then n^{th} derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ at the left side.

Examples of the Fredholm integro differential equations are given by:

$$a)\ u'(x) = 1 - \frac{1}{3}x + \int_0^1 xu(t)dt, \quad u(0) = 0,$$

$$b) u''(x) + u'(x) = x - sinx - \int_0^{\frac{\pi}{2}} xtu(t)dt, \ u(0) = 0, \ u'(0) = 1$$

Volterra Integro-Differential Equations

Volterra integro-differential equations appear when we convert initial value problems to integral equations. The Volterra integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x), \ \ n \geq 1$ inside and outside the integral sign. At least one of the limits of integration in this case is a variable as in the Volterra integral equations. The equation is called integro-differential because differential and integral operators are involved in the same equation. It is important to note that initial conditions should be given for Volterra integro-differential equations to determine the particular solutions. The Volterra integro-differential equation appears in the form:

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt \qquad (1.29)$$

Where $u^{(n)}$ indicates then n^{th} derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ at the left side.

Examples of the Volterra integro differential equations are given by

$$a)\ u'(x) = -1 + \frac{1}{3}x^2 - xe^x - \int_0^x tu(t)dt, \quad u(0) = 0,$$

$$b) u''(x) + u'(x) = 1 - x(sinx + cosx) - \int_0^x tu(t)dt, \ u(0) = -1, \ u'(0) = 1$$

Linearity and Homogeneity

Integral equations and integro-differential equations fall into two other types of classifications according tolinearity and homogeneity concepts. These two concepts play a major role in the structure of the solutions. In what follows, we highlight the definitions of these concepts.

Linearity Concept

If the exponent of the unknown function $u(x)$inside the integral sign is one, the integral equation or the integro-differential equation is called linear [6]. If the unknown function $u(x)$ has exponent other than one, or if the equation contains nonlinear functions of $u(x)$, such as $e^u, sinhu, cosu, \ln(1+u)$, the integral equation or the integro-differential equation is called nonlinear. To explain this concept, we consider the equations:

Examples of linear Volterra and Fredholm integral equations:

$$a)\ u(x) = 1 - \int_0^x (x-t)u(t)dt$$

$$b)\ u(x) = 1 - \int_0^1 (x-t)u(t)dt$$

Examples of nonlinear Volterra integral equation and nonlinear Fredholm integro-differential equations:

$$a)\ u(x) = 1 + \int_0^x (1 + x - t)u^4(t)dt$$

$$b)\ u'(x) = 1 + \int_0^x xte^{u(t)}dt, \quad u(0) = 1$$

It is important to point out that linear equations, except Fredholm integral equations of the first kind, give a unique solution if such a solution exists. However, solution of nonlinear equation may not be unique. Nonlinear equations usually give more than one solution and it is not usually easy to handle.

Homogeneity Concept

Integral equations and integro-differential equations of the second kind are classified as homogeneous or inhomogeneous, if the function $f(x)$ in the second kind of Volterra or Fredholm integral equations or integro-differential equations is identically zero, the equation is called homogeneous. Otherwise, it is called inhomogeneous. Notice that this property holds for equations of the second kind only. To clarify this concept we consider the following equations:

Homogeneous equations:

$$a)\ u(x) = \int_0^x (1 - x - t)u^4(t)dt$$

$$b)\ u''(x) = \int_0^x xtu(t)dt, \qquad u(0) = 1,\ u'(0) = 0$$

Inhomogeneous equations:

$$a)\ u(x) = sinx + \int_0^x xtu(t)dt$$

$$b)\ u(x) = x + \int_0^1 (x - t)^2 u(t)dt$$

The first two equations are inhomogeneous because $f(x) = sinx$and$f(x) = x$, whereas the last two equations are homogeneous because $f(x) = 0$ for each equation. We usually use specific approaches for homogeneous equations, and other methods are used for inhomogeneous equations.

Volterra Integral Equations

Volterra integral equations arise in many scientific applications such as the population dynamics spread of epidemics, and semi-conductor devices. We will also show that Volterra integral equations can be derived from initial value problems. Volterra started working on integral equations in 1884, but his serious study began in 1896. According to Bocher [1914], the name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908. Abel considered the problem of determining the equation of a curve in a vertical plane. In this problem, the time taken by a mass point to slide under the influence of gravity along this curve, from a given positive height, to the horizontal axis is equal to a prescribed function of the height. Abel derived the singular Abel's integral equation, a specific kind of Volterra integral equation that will study in this research, nonlinear Volterra integro-differential equation. Volterra integral equations of the first kind or the second kind, are characterized by a variable upper limit of integration[8].

For the first kind Volterra integral equations, the unknown function occurs only inside the integral sign in the form

$$f(x) = \int_0^x K(x,t)u(t)dt \qquad (2.1)$$

However, linear Volterra integral equations of the second kind, the unknown function $u(x)$occurs inside and outside the integral sign. The second kind Volterra integral equation is of the form

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt \qquad (2.2)$$

The kernel $K(x,t)$ and the function $f(x)$ are given real valued functions, and λ is a parameter

[9-10]. Our aim in this chapter is to study the linear Volterra integral equations of the second kind.

The Adomain Decomposition Method

The Adomain decomposition method (ADM) was introduced and developed by George Adomain in [11-12] and is well addressed in many references. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and non-linear ordinary differential equations, partial differential equations and integral equations as well.

The Adomain Decomposition method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (2.3)$$

Or, equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \ldots.. \qquad (2.4)$$

Where the components $u_n(x), n \geq 0$ are to be determined in a recursive manner. The decomposition method concefns itself with finding the components u_0, u_1, u_2 .... individually. The determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.

To obtain the recurrence relation, we substitute (2.4) into the Volterra integral equation (2.3) to get

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t)(\sum_{n=0}^{\infty} u_n(t))dt \qquad (2.5)$$

Or, equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_0^x K(x,t)[u_0(t) + u_1(t) + u_2(t) + \cdots]dt$$

(2.6)

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, the components $u_j(x), j \geq 1$ of the unknown function $u(x)$ are completely determined by setting the recurrence relation:

$$u_0(x) = f(x), \qquad u_{n+1}(x) = \lambda \int_0^x K(x,t)u_n(t)dt, \;\; n \geq 0 \qquad (2.7)$$

Which is equivalent to

$$u_0(x) = f(x),$$

$$u_1(x) = \lambda \int_0^x K(x,t)u_0(t)dt,$$

$$u_2(x) = \lambda \int_0^x K(x,t)u_1(t)dt, \qquad (2.8)$$

$u_3(x) = \lambda \int_0^x K(x,t)u_2(t)dt,$ …. and so on for other components.

The components $u_0(x), u_1(x), u_2(x), u_3(x)$, ... are completely determined. As a result, the solution $u(x)$of the Volterra integral equation (2.3) in a series form is readily obtained by using the series assumption in (2.4).

It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. It was formally shown by many researchers that if an exact solution exists for the problem, after the obtained series converges very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researches to confirm the rapid convergence of the resulting series. However, for concreted problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. The more components we consider the higher the accuracy exists.

The Variational Iteration Method

We study the another method called variation iteration method (VIM), that proved to be effective and reliable for analytic and numerical purposes. The variation iteration method (VIM), credit goes Ji-Huan He [13-14] is used to solve the linear, non-linear homogeneous and inhomogeneous equations. This method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists and not components as in Adomian decomposition method (ADM). The variational iteration method (VIM) handles both the linear and nonlinear provides in the same procedure without any need to specific conditions, such as the Adomian polynomials that we need for nonlinear problems. Moreover, this method gives the solution in a series form that converges to the closed form solution if an exact solution exists. The obtained series can be employed for numerical purposes if exact solution is not available. In that case, we present the main procedure of this method.

Consider the differential equation

$$Lu + Nu = g(t) \qquad (2.9)$$

Where L and N are linear and nonlinear operators respectively and g (t) is the source inhomogeneous term.

The variational iteration method presents a correction functional for equation (2.9) in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\big(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\big)d\xi \qquad (2.10)$$

Where λ is a general Lagrange's multiplier, nothing that in this method λ may be a constant or a function, and $\tilde{u}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{u}_n = 0,$ where δ is the variational derivative.

For a complete us of the variational iteration method, we should follow two steps, namely:

1. The determination of the Lagrange multiplier $\lambda(\xi)$ that will be identified optimally, and
2. With λ determined, we substitute the result into (2.10) where the restrictions should be omitted.

Taking the variation of (2.10) with respect to the independent variable u_n we find,

$$\frac{\delta u_{n+1}}{\delta u_n} = 1 + \frac{\delta}{\delta u_n}\left(\int_0^x \lambda(\xi)\big(Lu_n(\xi) + N\breve{u}_n(\xi) - g(\xi)\big)d\xi\right) \qquad (2.11)$$

Or equivalently,

$$\delta u_{n+1} = \delta u_n + \delta\big(\int_0^x \lambda(\xi)\big(Lu_n(\xi)\big)d\xi\big) \qquad (2.12)$$

For the determination of the Lagrange multiplier $\lambda(\xi)$, we use the integration by parts.

In other words, we can use

$$\int_0^x \lambda(\xi)\, u_n'(\xi)d\xi = \lambda(\xi)u_n(\xi) - \int_0^x \lambda'(\xi)u_n(\xi)\, d\xi,$$

$$\int_0^x \lambda(\xi)\, u_n''(\xi)d\xi = \lambda(\xi)u_n'(\xi) - \lambda'(\xi)u_n(\xi) + \int_0^x \lambda''(\xi)u_n(\xi)\, d\xi,$$

$$\int_0^x \lambda(\xi)\, u_n'''(\xi)d\xi = \lambda(\xi)u_n''(\xi) - \lambda'(\xi)u_n'(\xi) + \lambda''(\xi)u_n(\xi) - \int_0^x \lambda'''(\xi)u_n(\xi)\, d\xi,$$

$$\int_0^x \lambda(\xi)\, u_n^{iv}(\xi)d\xi = \lambda(\xi)u_n'''(\xi) - \lambda'(\xi)u_n''(\xi) + \lambda''(\xi)u_n'(\xi) - \lambda'''(\xi)u_n(\xi) + \int_0^x \lambda^{iv}(\xi)u_n(\xi)\, d\xi, \qquad (2.13)$$

and so on. The above identities are obtained by integrating by parts.

The successive approximations $u_{n+1}(x), n \geq 0$ of the solution of $u(x)$ will be readily obtained upon using selective function $u_0(x)$. However, for fast convergence, the function $u_0(x)$ should be selected by using the initial conditions as follows:

$u_0(x) = u(0),$ for the first order u'_n

$u_0(x) = u(0) + xu'(0)$ for the second order u''_n (2.14)

$u_0(x) = u(0) + xu'(0) + \frac{1}{2!}x^2u''(0)$ for the third order u'''_n

At last, the solution is given by $u = \lim_{n\to\infty} u_n$.

Iteration Formulas

Here, we have the iteration formulas for certain class of differential equations, its corresponding Lagrange multipliers, and its correction functional respectively:

$$(I)\begin{cases} u' + f(u(\xi), u'(\xi)) = 0, \lambda = -1 \\ u_{n+1} = u_n - \int_0^x [u_n' + f(u_n, u_n')]d\xi, \end{cases}$$

$$(II)\begin{cases} u'' + f(u(\xi), u'(\xi), u''(\xi)) = 0, \qquad \lambda = (\xi - x), \\ u_{n+1} = u_n + \int_0^x (\xi - x)\left[u_n'' + f\left(u_n, u_n', u_n''\right)\right]d\xi, \end{cases}$$

$$(III)\begin{cases} u''' + f(u(\xi), u'(\xi), u''(\xi), u'''(\xi)) = 0,\ \lambda = -\frac{1}{2!}(\xi - x)^2, \\ u_{n+1} = u_n - \int_0^x \frac{1}{2!}(\xi - x)^2[u_n''' + f(u_n, u_n', u_n'', u_n''')]d\xi, \end{cases} \quad (2.15)$$

$$(IV)\begin{cases} u^{iv} + f\left(u(\xi), u'(\xi), u''(\xi), u'''(\xi), u^{iv}(\xi)\right) = 0,\ \lambda = \frac{1}{3!}(\xi - x)^3, \\ u_{n+1}(x) = u_n + \int_0^x \frac{1}{3!}(\xi - x)^3[u_n''' + f\big(u_n, u_n', u_n'', u_n''', u^{iv}\big)]d\xi, \end{cases}$$

and generally,

$$(V)\begin{cases} u^{(n)} + f\big(u(\xi), u'(\xi), u''(\xi), u'''(\xi), \ldots.. u^{(n)}(\xi)\big) = 0, \lambda = (-1)^n \dfrac{1}{(n-1)!}(\xi - x)^{(n-1)}, \\ u_{n+1} = u_n + (-1)^n \displaystyle\int_0^x \frac{1}{(n-1)!}(\xi - x)^{n-1}[u_n^{(n)} + f\big(u_n, u_n', u_n'', u_n''', \ldots\ldots, u_n{}^{(n)}\big)]d\xi \end{cases}$$

For $n \geq 1$.

2.5 Solve the Volterra integral equation by using the Adomain decomposition method (ADM) and Variational iteration method (VIM). $\boldsymbol{u(x) = 1 + \int_0^x u(t)dt}$

Solution by ADM

We see that $f(x) = 1, \lambda = 1, K(x,t) = 1$.

Substituting the above values in $u(x) = \sum_{n=0}^{\infty} u_n(x)$ gives $\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x \sum_{n=0}^{\infty} u(t)dt$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 + \int_0^x [u_0(t) + u_1(t) + u_2(t) + \cdots]dt$$

Now, we set the following the recurrence relation.

$u_0(x) = 1, \quad u_{k+1}(x) = \int_0^x u_k(t)dt,\ \ k \geq 0$ that gives

$u_0(x) = 1,$

$u_1(x) = \int_0^x u_0(t)dt = \int_0^x dt = \frac{x^2}{2!},$

$u_2(x) = \int_0^x u_1(t)dt = \int_0^x \frac{t^2}{2} dt = \frac{x^3}{6} = \frac{x^3}{3!},$

$u_3(x) = \int_0^x u_2(t)dt = \int_0^x \frac{t^3}{6} dt = \frac{x^4}{24} = \frac{x^4}{4!},$

$u_4(x) = \int_0^x u_3(t)dt = \int_0^x \frac{t^4}{24} dt = \frac{x^5}{120} = \frac{x^5}{5!},$

$u_5(x) = \int_0^x u_4(t)dt = \int_0^x \frac{t^5}{120} dt = \frac{x^6}{720} = \frac{x^6}{6!},$

………………………………………………..

$u_{k+1}(x) = \int_0^x u_k(t)dt, \ \ k \geq 0,$

The solution in a series form is given by

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots ....$$

and in a closed form by $u(x) = e^x$

Solution by VIM

$$\boldsymbol{u(x) = 1 + \int_0^x u(t)dt} \qquad (2.15)$$

Using Leobnitz rule to differentiate both sides of (2.15) gives

$$u'(x) - u(x) = 0 \qquad (2.16)$$

Substituting $x = 0$ into (2.15) gives the initial condition $u(0) = 1.$

The correction functional for equation (2.15) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)(u_n'(\xi) - \tilde{u}_n(\xi))d\xi \qquad (2.17)$$

Using the iteration formula

$$\begin{cases} u' + f(u(\xi), u'(\xi)) = 0, \lambda = -1 \\ u_{n+1} = u_n - \int_0^x [u_n' + f(u_n, u_n')]d\xi, \end{cases} \qquad (2.18)$$

gives the value of $\lambda = -1.$

Substituting the value of the Larange multiplier $\lambda = -1$ into (1.14) gives the iteration formula

$$u_{n+1}(x) = u_n(x) - \int_0^x (u_n'(\xi) - u_n(\xi))d\xi \qquad (2.19)$$

We use the initial condition to select $u_0(x) = u(0) = 1$.

Using the initial conditions into (1.16) gives the following successive approximation.

$$u_0(x) = 1$$

$$u_1(x) = 1 - \int_0^x (u_0'(\xi) - u_0(\xi))d\xi = 1 + x$$

$$u_2(x) = 1 + x - \int_0^x (u_1'(\xi) - u_1(\xi))d\xi = 1 + x + \frac{1}{2!}x^2$$

$$u_3(x) = 1 + x + \frac{1}{2!}x^2 - \int_0^x (u_2'(\xi) - u_2(\xi))d\xi = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$

$$u_4(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \int_0^x (u_3'(\xi) - u_3(\xi))d\xi$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

$$u_5(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \int_0^x (u_4'(\xi) - u_4(\xi))d\xi$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5$$

and so on.

The VIM admits the use of $u(x) = \lim_{n\to\infty} u_n(x)$

$$u(x) = \lim_{n\to\infty}\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \ldots + \frac{1}{n!}x^n\right)$$

That gives the exact solution by $u(x) = e^x$

X	$EXACT$	ADM	$VIM(u_0(x$	$VIM(u_1(x)$	$VIM(u_2(x)$	$VIM(u_3(x))$	$VIM(u_4(x))$	$VIM(u_5(x))$
0.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

0.4	1.491820	1.491820	1.000000	1.400000	1.480000	1.490670	1.491730	1.491820
0.8	2.225540	2.225490	1.000000	1.800000	2.120000	2.205330	2.222400	2.225130
1.2	3.320120	3.319280	1.000000	2.200000	2.920000	3.208000	3.294400	3.315140
1.6	4.953030	4.946420	1.000000	2.600000	3.880000	4.562670	4.835730	4.923110
2.0	7.389060	7.355560	1.000000	3.000000	5.000000	6.333330	7.000000	7.266670
2.4	11.023200	10.895400	1.000000	3.400000	6.280000	8.584000	9.966400	10.630000
2.8	16.444600	16.043200	1.000000	3.800000	7.720000	11.378700	13.939700	15.373900
3.0	20.085500	19.412500	1.000000	4.000000	8.500000	13.000000	16.375000	18.400000

Table: 2.5. 1 Comparison between the ADM solutions with VIM solutions using five iterations for

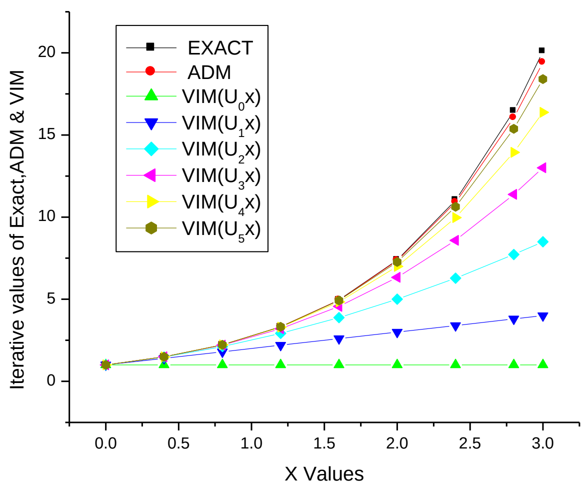

Fig: 2.5.2 Comparison between the Exact solutions with ADM & VIM solutions using five iterations

In Figure 2.5.2, it was noted that the exact values coincide with ADM. But, for lower iterates of $VIM(u_0(x))$, the values are not coincide with exact values, whereas $VIM(u_1(x)), VIM(u_2(x)), VIM(u_3(x)), VIM(u_4(x))$, and $VIM(u_5(x))$ values approaching to reach the exact value. But we have observed that the higher components of VIM, coincides with the exact value as well as with the ADM components.

1.5 Solve the Volterra integral equation by using the Adomain decomposition method (ADM) and Variational iteration method (VIM).

2. $\mathrm{u(x) = x + \int_0^x (x-t)u(t)dt}$

Solution by ADM

We see that $f(x) = x, \lambda = 1, K(x,t) = (x-t)$

Substituting the above values in $u(x) = \sum_{n=0}^{\infty} u_n(x)$ gives

$$\sum_{n=0}^{\infty} u_n(x) = x + \int_0^x \sum_{n=0}^{\infty} (x-t)u(t)dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = x + \int_0^x (x-t)[u_0(t) + u_1(t) + u_2(t) + \cdots]dt$$

Now, we set the following the recurrence relation.

$u_0(x) = x,$

$u_{k+1}(x) = \int_0^x (x-t)u_k(t)dt, \ \ k \geq 0$ that gives

$u_0(x) = x,$

$$u_1(x) = \int_0^x (x-t)u_0(t)dt = \int_0^x (x-t)tdt = \frac{x^3}{6} = \frac{x^3}{3!},$$

$$u_2(x) = \int_0^x (x-t)u_1(t)dt = \int_0^x (x-t)\frac{t^3}{6}dt = \frac{x^5}{120} = \frac{x^5}{5!},$$

$$u_3(x) = \int_0^x (x-t)u_2(t)dt = \int_0^x (x-t)\frac{t^5}{120}dt = \frac{x^7}{5040} = \frac{x^7}{7!},$$

$$u_4(x) = \int_0^x (x-t)u_3(t)dt = \int_0^x (x-t)\frac{t^7}{5040}dt = \frac{x^9}{362880} = \frac{x^9}{9!},$$

$$u_5(x) = \int_0^x (x-t)u_4(t)dt = \int_0^x (x-t)\frac{t^9}{362880}dt = \frac{x^{11}}{39916800} = \frac{x^{11}}{11!},$$

…………………………………………………………………………………..

$u_{k+1}(x) = \int_0^x u_k(t)dt, \;\; k \geq 0,$

The solution in a series form is given by

$$u(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \cdots ....$$

and in a closed form by $u(x) = Sinh[x]$

Solution by VIM

$$\boldsymbol{u(x) = x + \int_0^x (x-t)u(t)dt} \qquad (2.20)$$

Using Leibnitz rule to differentiate both sides of (2.20) with respect to x gives the equation

$$u'(x) = 1 + \int_0^x u(t)dt, \quad u(0) = 0 \qquad (2.21)$$

Differentiating (1.18) with respect to x we get the differential equation

$$u''(x) = u(x) \qquad (2.22)$$

Substituting $x = 0$ into (2.20) and (2.22) gives the initial condition $u(0) = 0; u'(0) = 1$.

The resulting initial value problem that consists of a second order ODE and initial conditions is given by

$$u''(x) - u(x) = 0; \;\; u(0) = 0; \; u'(0) = 1 \qquad (2.23)$$

The integro-differential equation (2.21) and the initial value problem (2.23) will be solved using the variational iteration method.

The correction functional for (2.21) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)(u_n''(\xi) - \tilde{u}_n(\xi))d\xi \tag{2.24}$$

Using the iteration formula

$$\begin{cases} u'' + f(u(\xi), u'(\xi), u''(\xi)) = 0, \ \lambda = (\xi - x), \\ u_{n+1} = u_n + \int_0^x (\xi - x)\left[u_n'' + f\left(u_n, u_n', u_n''\right)\right]d\xi, \end{cases} \tag{2.25}$$

gives the value of $\lambda = (\xi - x)$.

Substituting the value of the Larange multiplier $\lambda = (\xi - x)$ into (2.24) gives the iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x (\xi - x)\left(u_n''(\xi) - \tilde{u}_n(\xi)\right)d\xi \tag{2.26}$$

We use the initial condition to select $u_0(x) = u(0) + xu'(0) = x$. Using this into (2.24) gives the following successive approximations:

$u_0(x) = x,$

$$u_1(x) = x + \int_0^x (\xi - x)\left(u_0''(\xi) - u_0(\xi)\right)d\xi = x + \frac{x^3}{3!},$$

$$u_2(x) = x + \frac{x^3}{3!} + \int_0^x (\xi - x)\left(u_1''(\xi) - u_1(\xi)\right)d\xi = x + \frac{x^3}{3!} + \frac{x^5}{5!},$$

$$u_3(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \int_0^x (\xi - x)\left(u_2''(\xi) - u_2(\xi)\right)d\xi = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!},$$

$$u_4(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \int_0^x \left(u_3''(\xi) - u_3(\xi)\right)d\xi = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!},$$

$$u_5(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \int_0^x \left(u_4''(\xi) - u_4(\xi)\right)d\xi = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!},$$

………………………………………………………………………………..

$$u_n(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \cdots \ldots + \frac{1}{(2n+1)!}x^{2n+1}$$

The VIM admits the use of $u(x) = \lim_{n\to\infty} u_n(x)$

$$u(x) = \lim_{n\to\infty}(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \cdots \ldots + \frac{1}{(2n+1)!}x^{2n+1})$$

Which gives the exact solution by $u(x) = Sinh[x]$

X	$EXACT$	ADM	$VIM(u_0(x)$	$VIM(u_1(x))$	$VIM(u_2(x))$	$VIM(u_3(x))$	$VIM(u_4(x))$	$VIM(u_5(x))$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.4	0.410752	0.410752	0.400000	0.410667	0.410752	0.410752	0.410752	0.410752
0.8	0.888106	0.888106	0.800000	0.885333	0.888064	0.888106	0.888106	0.888106
1.2	1.509460	1.509460	1.200000	1.488000	1.580740	1.509450	1.509460	1.509460
1.6	2.375570	2.375570	1.600000	2.282670	2.370050	2.375370	2.375560	2.375570
2.0	3.626860	3.626860	2.00000	3.333330	3.600000	3.625400	3.626810	3.626860
2.4	5.466230	5.466210	2.400000	4.704000	5.367550	5.458550	5.465830	5.466210
2.8	8.191920	8.191810	2.800000	6.458970	7.892860	8.160580	8.189730	8.191810
3.0	10.017900	10.017600	3.00000	7.500000	9.525000	9.958930	10.01320	10.017600

Table: 2. 6.1 Comparison between the ADM solutions with VIM solutions using five iterations for $0 < x < 1$

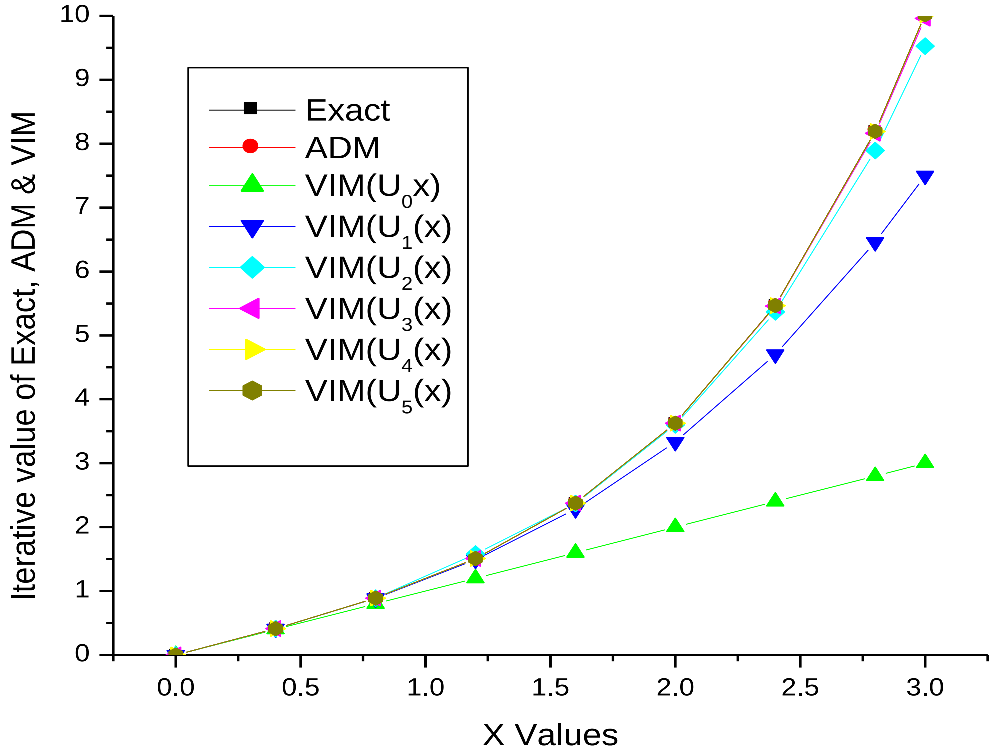

Fig: 2.6.2 Comparison between the Exact solution with ADM& VIM solutions using five iterations

In Figure 2.6.2 it was noted that the scheme yields highly accurate solution after $x = 1.0$ and coincides with the exact and ADM values. It was observed that for higher iterates of ADM and VIM the scheme is getting closer to the exact value and gives the efficiency results as compared to that of the lower components of VIM.

Conclusion

In this work, we solved the linear Volterra integral equations of second kind using the Adomain Decomposition method and Variational Iteration method. Both the methods give the efficient results for higher components of ADM and VIM, which are near to the exact values of the given function, which is depicted in Figures (2.5.2) and (2.6.2).

Fredholm Integral Equations

The Fredholm integral equations arise in many scientific applications such as in the theory of signal processing and in physics, the solution of such integral equations allows experimental spectra to be related to various underlying distributions, for instance the mass distribution of polymers in a polymeric melt or the distribution of relaxation times in the system. It was shown that Fredholm integral equations could be derived from boundary value problems. Erik Ivar Fredholm (1866-1927) is best recognized by his work on integral equations and spectral theory. He was a Swedish mathematician who established the theory of integral equations and his 1903 paper in Acta Mathematica, which played a vital role in the development of operator theory. Unlike Volterra integral equations where at least one of the limits of integration is a variable, Fredholm integral equations are characterized by fixed limits of integration of the form [15–21]

For the first kind Fredholm integral equations, the unknown function $u(x)$ occurs only under the integral sign in the form

$$f(x) = \int_a^b K(x,t)u(t)dt \qquad (3.1)$$

Where a and b are constants.

However, the Fredholm integral equations of the second kind, the unknown function $u(x)$ occurs inside and outside the integral sign. The second kind is represented by the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (3.2)$$

The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions and λ is a parameter. Where a and b are constants.

The Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced and developed by George Adomian in [11-12]. The ADM consists of decomposing the unknown function $u(x)$of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (3.3)$$

Or equivalently, Or, equivalently $u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \dots ..$ (3.4)

Where the components $u_n(x), n \geq 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components u_0, u_1, u_2 .... individually. The determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.

To obtain the recurrence relation, we substitute (3.3) into the Fredholm integral equation of second kind (3.2) to get

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt \qquad (3.5)$$

Or equivalently,

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_a^b K(x,t)[u_0(t) + u_1(t) + u_2(t) + \cdots]dt$$
(3.6)

The zeroth component

$$u_0(x) = f(x)$$

$$u_1(x) = \lambda \int_a^b K(x,t) u_0(t) dt,$$

$u_2(x) = \lambda \int_a^b K(x,t)u_1(t)dt,$ (3.7)

$u_3(x) = \lambda \int_a^b K(x,t)u_2(t)dt,$

………………………………..

$$u_{n+1}(x) = \lambda \int_a^b K(x,t)u_n(t)dt, \qquad n \geq 0,$$

In view of (3.7), the components $u_0(x), u_1(x), u_2(x), u_3(x)$ …. are completely determined. As a result, the solution $u(x)$ of the Fredholm integral equation (3.2) is obtained in a series form by using the series in (3.3).

It was observed that the decomposition method converted the integral equation into an elegant determination of computable components. If an exact solution exists for the problem, after the obtained series converges very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researches to confirm the rapid convergence of the resulting series. However, for concreted problems, where a closed form solution is not obtainable, a truncated number of terms are usually used for numerical purposes. To get the accuracy of the solution we have to consider more components of the series.

The Variational Iteration Method

In this part, we apply the variational iteration method to the Fredholm integral equation. This method works effectively if the kernel $K(x,t)$ is separable and can be written in the form $K(x,t) = g(x)h(t)$. this approach to be used here is identical to the approach used in the earlier section. We should differentiate both sides of the Fredholm integral equation to convert it to an identical Fredholm integro-differential equation. It is important to note that integro-differential equation needs an initial condition that should be defined. Various kinds of analytical methods and numerical methods [22-23] were used to solve integral equations. We apply He's variational iteration method [24–28] to solve integral equations.

The standard Fredholm integral equation of second kind is of the form

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt \qquad (3.8)$$

Or equivalently $u(x) = f(x) + g(x)\int_a^b h(t)u(t)dt$ (3.9)

Differentiating on both sides of (2.9) with respect to x gives

$$u^{'}(x) = f^{'}(x) + g^{'}(x)\int_a^b h(t)u(t)dt \qquad (3.10)$$

The correction functional for the integro-differential equation (2.10) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n'(\xi) - f'(\xi) - g'(\xi)\int_a^b h(r)\tilde{u}_n(r)dr\right)d\xi \qquad (3.11)$$

As discussed in the earlier chapter, the variational iteration method is used by applying two essential steps. It is necessary first to determine the Lagrange multiplier $\lambda(\xi)$ that can be identified optimally via integration by parts and by using a restricted variation. However, $\lambda(\xi) = -1$, for first order integro-differential equations. Having determined $\lambda,$ an iteration formula, without restricted variation , given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - f'(\xi) - g'(\xi)\int_a^b h(r)u_n(r)dr\right)d\xi \qquad (3.12)$$

is used for the determination of the successive approximations

$u_{n+1}(x),\ n \geq 0$

of the solution $u(x)$. The zeroth approximation u_0 can be a selective function. However, using the given initial value $u(0)$ is preferably used for the selective zeroth approximation u_0.

Consequently, the solution is given by $u(x) = \lim_{n\to\infty} u_n(x)$.

3.4 Solve the Fredholm integral equation of second kind using Adomian decomposition method (ADM) and Variational iteration method(VIM)

$$\boldsymbol{u(x) = e^x - x + x\int_0^1 tu(t)dt}$$

Solution by ADM

The ADM assumes that the solution $u(x)$ has a series form given in $u(x) = \sum_{n=0}^{\infty} u_n(x)$
Substituting the decomposition series $u(x) = \sum_{n=0}^{\infty} u_n(x)$ into both sides of

$u(x) = e^x - x + x\int_0^1 tu(t)dt$ gives

$$\sum_{n=0}^{\infty} u_n(x) = u(x) = e^x - x + x\int_0^1 tu_n(t)dt \qquad (3.13)$$

Or equivalently

$$u_0(x) + u_1(x)+u_2(x) + \cdots = e^x - x + x\int_0^1 t[u_0(t) + u_1(t)+u_2(t) + \cdots]dt \qquad (3.14)$$

We identify the zeroth component by all terms that are not included under the integral sign.

Therefore, we obtain the following recurrence relation

$u_0(x) = e^x - x,$

$u_{k+1}(x) = x\int_0^1 tu_k(t)dt, \qquad k \geq 0.$

We obtain the following terms:

$u_0(x) = e^x - x,$

$$u_1(x) = x\int_0^1 tu_0(t)\,dt = x\int_0^1 (e^t - t)\,dt = \frac{2}{3}x,$$

$$u_2(x) = x\int_0^1 tu_1(t)\,dt = x\int_0^1 \frac{2}{3}t^2\,dt = \frac{2}{9}x,$$

$$u_3(x) = x\int_0^1 tu_2(t)\,dt = x\int_0^1 \frac{2}{9}t^2\,dt = \frac{2}{27}x, \qquad (3.15)$$

$$u_4(x) = x\int_0^1 tu_3(t)\,dt = x\int_0^1 \frac{2}{27}t^2\,dt = \frac{2}{81}x,$$

$$u_5(x) = x\int_0^1 tu_4(t)\,dt = x\int_0^1 \frac{2}{81}t^2\,dt = \frac{2}{243}x,$$

…………………………………………………..

And so on. Using the $u(x) = \sum_{n=0}^{\infty} u_n(x)$ gives the series solution

$$u(x) = e^x - x + \frac{2}{3}x\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots\right) \qquad (3.16)$$

Notice that the infinite geometric series at the right hand side has $a = 1,$ and the ratio $r = \frac{1}{3}$. .

The sum of the infinite series is given by $s = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

$$u(x) = e^x - x + \frac{2}{3}x\left(\frac{3}{2}\right) = e^x - x + x = e^x$$

The series solution of (3.16) converges to the closed form solution $u(x) = e^x$.

Solution by VIM

$$\boldsymbol{u(x) = e^x - x + x\int_0^1 tu(t)dt} \qquad (3.17)$$

Differentiating both sides of (3.17) with respect to x yields,

$$u'(x) = e^x - 1 + \int_0^1 tu(t)dt \qquad (3.18)$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\xi) - e^{\xi} + 1 - \int_0^1 ru_n(r)dr\right) d\xi \qquad (3.19)$$

Here we use $\lambda = -1$, for first order integro-differential equations. The initial condition $u(0) = 1$, is obtained by substituting $x = 0$, into (3.17).

Therefore $u_0(x) = u(0) = 1$.

Using this into the correctional functional (3.19) gives the following successive approximations.

$$u_0(x) = 1,$$

$$u_1(x) = u_0(x) - \int_0^x \left(u_0'(\xi) - e^{\xi} + 1 - \int_0^1 ru_0(r)dr\right) d\xi = e^x - \frac{1}{2}x,$$

$$u_2(x) = u_1(x) - \int_0^x \left(u_1'(\xi) - e^{\xi} + 1 - \int_0^1 ru_1(r)dr\right) d\xi = e^x - \frac{1}{2.3}x,$$

$$u_3(x) = u_2(x) - \int_0^x \left(u_2'(\xi) - e^{\xi} + 1 - \int_0^1 ru_2(r)dr\right) d\xi = e^x - \frac{1}{2.3^2}x,$$

$$u_4(x) = u_3(x) - \int_0^x \left(u_3'(\xi) - e^{\xi} + 1 - \int_0^1 ru_3(r)dr\right) d\xi = e^x - \frac{1}{2.3^3}x,$$

$$u_5(x) = u_4(x) - \int_0^x \left(u_4'(\xi) - e^{\xi} + 1 - \int_0^1 ru_4(r)dr\right) d\xi = e^x - \frac{1}{2.3^4}x,$$

…………………………………………………………………………………

$$u_{n+1}(x) = e^x - \frac{1}{2.3^n}x, \;\; n \geq 0.$$

The VIM confirms the use of $u(x) = \lim_{n\to\infty} u_n(x) = e^x$.

X	$EXACT$	ADM	$VIM(u_0(x)$	$VIM(u_1(x))$	$VIM(u_2(x))$	$VIM(u_3(x))$	$VIM(u_4(x))$	$VIM(u_5(x))$
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.1	1.10517	1.10476	1.00000	1.05517	1.08850	1.09962	1.10332	1.10476
0.2	1.22140	1.22058	1.00000	1.12140	1.18807	1.21029	1.21770	1.22058
0.3	1.34986	1.34862	1.00000	1.19986	1.29986	1.33319	1.34430	1.34862
0.4	1.49182	1.49018	1.00000	1.29182	1.42516	1.46960	1.48442	1.49018
0.5	1.64872	1.64666	1.00000	1.39872	1.56539	1.62094	1.63946	1.64666
0.6	1.82212	1.81965	1.00000	1.52212	1.72212	1.78879	1.81101	1.81965
0.7	2.01375	2.01087	1.00000	1.66375	1.89709	1.97486	2.00079	2.01087
0.8	2.22554	2.22225	1.00000	1.82554	2.09221	2.18110	2.21073	2.22225
0.9	2.45960	2.45590	1.00000	2.00960	2.30960	2.4096	2.44294	2.45590
1.0	2.71828	2.71417	1.00000	2.21828	2.55162	2.66273	2.69976	2.71417

Table: 3.4.1 Comparison of ADM and VIM solutions using five iterations with Exact Values

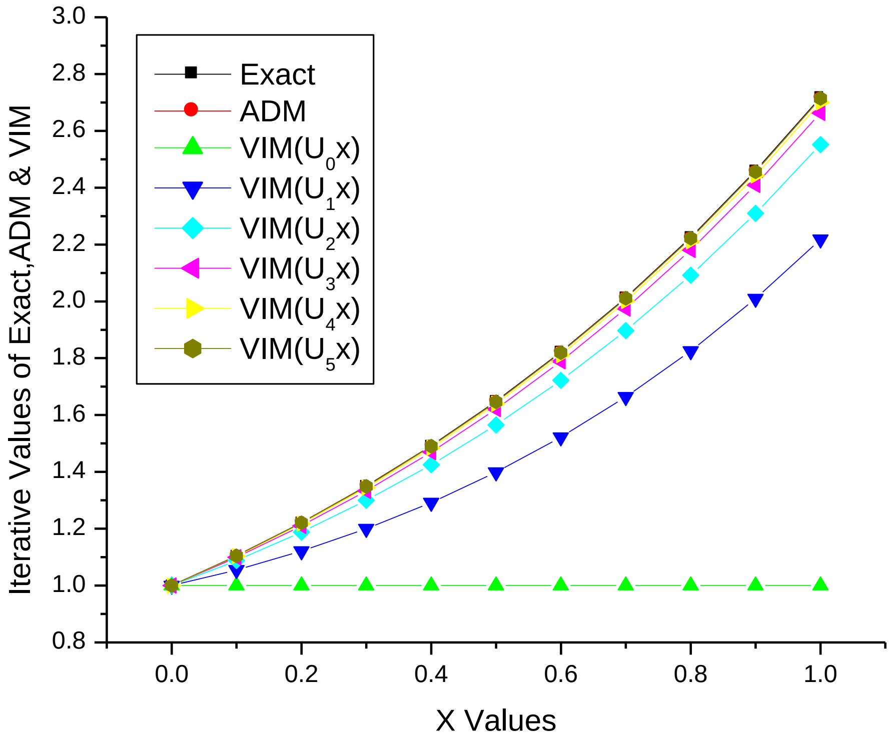

Fig: 3.4.2 Comparison between the Exact solution with ADM & VIM solutions using five iterations

The numeric values of ADM and VIM for five iterates are depicted in the Figure 3.4.2. It was noted that the values of higher components of VIM overlapping with the values of exact and ADM. It shows that there is a more error between lower components of VIM with the values of ADM and exact. For example, the zeroth component $(VIMu_0(x))$ is linear to $x - axis$. but as the value of the components increases, it is approaching the value of exact and ADM.

3.5 Solve the Fredholm integral equation of second kind using Adomian decompositon method (ADM) and Variational iteration method(VIM)

$$\boldsymbol{u(x) = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u(t)dt}$$

Solution by ADM

The ADM assumes that the solution $u(x)$ has a series form given in $u(x) = \sum_{n=0}^{\infty} u_n(x)$

Substituting the decomposition series $u(x) = \sum_{n=0}^{\infty} u_n(x)$ into bothsides of

$u(x) = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u(t)dt$ gives

$$\sum_{n=0}^{\infty} u_n(x) = u(x) = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u(t)dt \qquad (3.20)$$

Or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)[u_0(t) + u_1(t) + u_2(t) + \cdots]dt$$
(3.21)

We identify the zeroth component by all terms that are not included under the integral sign.

Therefore, we obtain the following recurrence relation

$u_0(x) = \cos(x),$

$$u_{k+1}(x) = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_k(t)dt\,, \qquad k \geq 0$$

We obtain the following terms:

$u_0(x) = \cos(x),$

$$u_1(x) = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_0(t)dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)\cos(t)dt = \frac{\sin(x)}{2},$$

$$u_2(x) = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_1(t)dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)\frac{\sin(t)}{2}dt = \frac{\sin(x)}{4},$$

$$u_3(x) = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_2(t)dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)\frac{\sin(t)}{4}dt = \frac{\sin(x)}{8},$$

$$u_4(x) = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_3(t)dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)\frac{\sin(t)}{8}dt = \frac{\sin(x)}{16},$$

$$u_5(x) = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_4(t)dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)\frac{\sin(t)}{16}dt = \frac{\sin(x)}{32},$$

…………………………………………………………………

And so on. Using the $u(x) = \sum_{n=0}^{\infty} u_n(x)$ gives the series solution

$$u(x) = \cos(x) + \sin(x)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \ldots\right) \quad (3.22)$$

$$u(x) = \cos(x) + \sin(x)\left(1\left(\frac{1}{2^0}\right) + \frac{1}{2}\left(\frac{1}{2^1}\right) + \frac{1}{2}\left(\frac{1}{2^2}\right) + \frac{1}{2}\left(\frac{1}{2^3}\right) + \frac{1}{2}\left(\frac{1}{2^4}\right) + \cdots \ldots\right)$$

$$u(x) = \cos(x) + \sin(x)\left(\frac{1}{2}\left(\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \ldots\right)\right)$$

Notice that the infinite geometric series at the right hand side has $a = 1,$ and the ratio $r = \frac{1}{2}$.

The sum of the infinite series is given by $s = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

$$u(x) = \cos(x) + \sin(x)\left(\frac{1}{2}(2)\right) = \cos(x) + \sin(x)$$

The series solution of (2.22) converges to the closed form solution $u(x) = \cos(x) + \sin(x)$.

Solution by VIM

$$u(x) = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u(t)dt \quad (3.23)$$

The correction functional for (2.23) is given by

$$u_{n+1}(x) = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_n(t)dt \quad (3.24)$$

Therefore, the initial approximation for (3.23) is $u_0(x) = \cos(x)$.

Using this into the correctional functional (3.24) gives the following successive approximations.

$u_0(x) = \cos(x),$

$u_1(x) = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_0(t)dt = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)cos(t)dt = \cos(x) + \frac{1}{2}\sin(x)$

$u_2(x) = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_1(t)dt = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)\left(\cos(t) + \frac{1}{2}\sin(t)\right)dt = \cos(x) + \frac{3}{4}\sin(x) = \cos(x) + \left(\frac{2^2-1}{2^2}\right)\sin(x)$

$$u_3(x) = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_2(t)dt = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)\left(\cos(t) + \frac{3}{4}\sin(t)\right)dt$$

$$= \cos(x) + \frac{7}{8}\sin(x) = \cos(x) + \left(\frac{2^3-1}{2^3}\right)\sin(x)$$

$$u_4(x) = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_3(t)dt = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)\left(\cos(t) + \frac{7}{8}\sin(t)\right)dt$$

$$= \cos(x) + \frac{15}{16}\sin(x) = \cos(x) + \left(\frac{2^4-1}{2^4}\right)\sin(x)$$

$$u_5(x) = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)u_4(t)dt = cos(x) + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)\left(\cos(t) + \frac{15}{16}\sin(t)\right)dt$$

$$= \cos(x) + \frac{31}{32}\sin(x) = \cos(x) + \left(\frac{2^5 - 1}{2^5}\right)\sin(x)$$

……………………………………………………………………………………

Continuing like this, we obtain

$$u(x) = \cos(x) + \lim_{n\to\infty}\left(\frac{2^n - 1}{2^n}\right)\sin(x) = \cos(x) + \sin(x)$$

which is the exact solution $u(x) = \cos(x) + \sin(x)$.

X	$EXACT$	ADM	$VIM(u_0(x)$	$VIM(u_1(x))$	$VIM(u_2(x))$	$VIM(u_3(x))$	$VIM(u_4(x))$	$VIM(u_5(x))$
0.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	1.094840	1.091720	0.995004	1.044920	1.069880	1.082360	1.088600	1.091720
0.2	1.178740	1.172530	0.980067	1.079400	1.129070	1.153900	1.166320	1.172530
0.3	1.250860	1.241620	0.955336	1.103100	1.176980	1.213920	1.232390	1.241620
0.4	1.310480	1.298310	0.921061	1.115770	1.213120	1.261800	1.286140	1.298310
0.5	1.357010	1.342030	0.877583	1.117300	1.237150	1.297080	1.327040	1.342030
0.6	1.389980	1.372330	0.825336	1.107660	1.248820	1.319400	1.354690	1.372330
0.7	1.409060	1.388930	0.764842	1.086950	1.248010	1.328530	1.368800	1.388930
0.8	1.414060	1.391650	0.696707	1.055380	1.234720	1.324390	1.369230	1.391650
0.9	1.404940	1.380460	0.621610	1.013270	1.209110	1.307020	1.355980	1.380460
1.0	1.381770	1.355480	0.540302	0.961038	1.171410	1.276590	1.329180	1.355480

Table: 3.5.1 Comparison of ADM and VIM solutions using five iterations with Exact Values

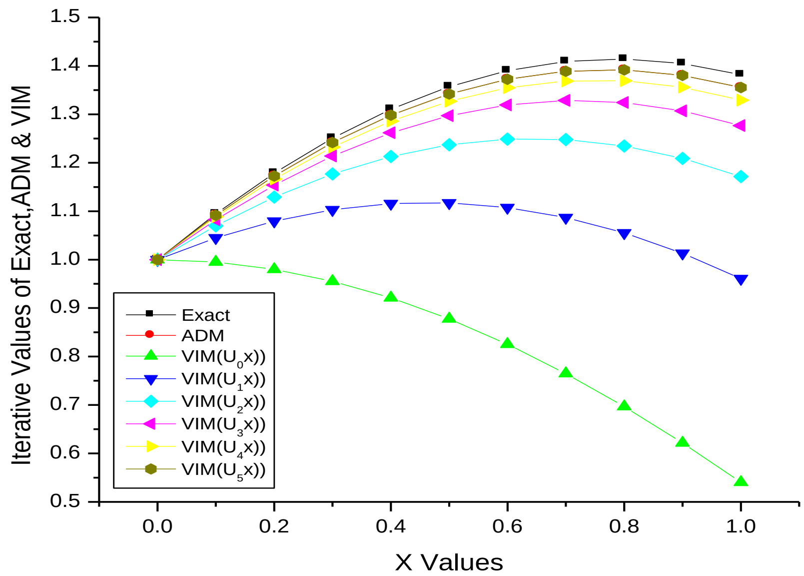

Fig: 3.5.2 Comparison between the Exact solution with ADM& VIM solutions using five iterations

The numeric values of ADM and VIM for five iterates are depicted in the Figure 3.5.2 It was noted that the ADM and exact values coincides with each other. The error between lower components of VIM is very large as compared to that for the higher components of VIM. It has depicted for the zeroth component and the first component the values decreased below the value of one, whereas the higher components approaching the value of exact and ADM.

Conclusion

In this work, we solved the linear Fredholm integral equations of second kind, using the Adomain Decomposition method and Variational Iteration method. Both the methods give the efficient results, which are near to the exact values of the given function, which is depicted in Figures (3.4.2) and (3.5.2). The error between the exact values and the values of

ADM and VIM, is also very negligible as compared to that of the lower components of VIM

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