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# Formal Power Series
# The Lagrange- Burmann Theo

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## Abstract:

This research was used the algebraic properties of the formal power series (fps) as well as the definition of formal Laurent series (fLs) to understand the deep thoughts of the Lagrange- Burmann Theorem. The aim of this study is using the proof of Lagrange-Burmann theory to find the solution to the following problem:

Let $F \coloneqq x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots$ ... ... and $P \coloneqq \frac{2x}{1-x^2}$ , $Q = P^{[-1]}$. Prove the functional relation $FoQ = \frac{1}{2}\ F$ . In conclusion, we can prove it by showing that $F = \frac{1}{2}\ FoP$ .

**Keywords:** *formal power series (fps), formal Laurent series (fls), Lagrange- Burmann theorem.*

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# Introduction

Formal power series is a sequence of polynomials which contains a infinite numbers of terms. Thus, there are many of variables which have arbitrary numbers. In fact, the possibility of replacing these variables are so difficult. In these papers, I will present the definition and the properties of formal power series as well as the composition and differentiation of the formal power series. It illustrates the Lagrange- Burmann theorem by including a proof and a problem as well.

**Algebraic Preliminaries: Complex Numbers:**

- Definition: a rule of composition is defined in $\mathfrak{I}$ as $\forall(a,b) \in \mathfrak{I}$ there is a unique product of a and b which denoted by $a * b$. Further, it is not necessarily that $a * b = b * a$.
- Associative: $(a * b) * c = a * (b * c)$ for all $a, b, c \in \mathfrak{I}$.
- Identity and inverse: $\exists e \in \mathfrak{I}$ such that $e * a = a,\ \forall a \in \mathfrak{I}$. Moreover, $\forall a \in \mathfrak{I}, \exists \bar{a} \in \mathfrak{I}$ such that $\bar{a} * a = e$.
- Commutative law: in $\mathfrak{I}$ , the operation $*$ holds because $a * b = b * a$ for all $a, b \in \mathfrak{I}$. Hence, $\mathfrak{I}$ is an abelian group.
- Semigroup: if a math system holds (a rule of composition, associative, and inverse) axioms, it is called semigroup. For example, the square matrices with real elements under the operation of multiplication form a semigroup.
- The field of complex numbers: the complex numbers $\mathbb{C}$ always introduced by the pair $(\alpha, \beta)$ where $\alpha, \beta \in \mathbb{R}$. Addition operation is defined by $(\alpha, \beta) + (\gamma, \delta) = (\alpha + \gamma, \beta + \delta)$. With respect to the multiplication, the identity is (0,0) and the inverse is $(-\alpha, -\beta)$. The multiplication operation is defined by $(\alpha, \beta).(\gamma, \delta) = (\alpha\gamma - \beta\delta, \alpha\delta + \beta\gamma)$. The identity element is (1,0) and the inverse is not equal to (0,0). In two operations additive and multiplication, it is clear that the associative axiom holds.

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- Let $z = (\alpha, \beta) = \alpha + i\beta$ a complex number, then $|z| = (\alpha^2 + \beta^2)^{\frac{1}{2}}$, whereas $\alpha = |z| \cos \emptyset, \beta = |z| \sin \emptyset$.
- Complex numbers and triangle inequality: $|z_1 + z_2| \le |z_1| + |z_2|$. Indeed,

$$|z|^2 = z\bar{z} \Rightarrow |z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$$
$$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$
$$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$$
$$= |z_1|^2 + 2Rez_1\overline{z_2} + |z_2|^2 \quad , Rez = \frac{z+\bar{z}}{2}$$
$$\le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$
$$= (|z_1| + |z_2|)^2$$

**Definition and algebraic properties of formal power series:**

A formal power series (fps) is an infinite sequence $\{a_0, a_1, a_2, \dots \dots, a_n\}$ of elements in the field F. In other words, it is a function from the positive integers in F such that $\{0,1,2,3, \dots \dots\} \to F$. It is always written as $P = a_0 + a_1x + a_2x^2 + \cdots \dots$ .

- Example: $1 + 1!\,x + 2!\,x^2 + \cdots \dots$ . Moreover, Cauchy product of two formal power series: $PQ = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots \dots \dots$ .
- Theorem: the formal power series over F form an integral domain. The unite of the integral domain $P = a_0 + a_1x + \cdots \dots$ over F are the series with $a_0 \neq 0$.
- The inverse of (fps): the inverse with respect to multiplication of a fps P is the reciprocal of P, $P^{-1}$ whereas the inverse with respect to addition is $-P$.
- Remark: if $a_0 = 0$, then according to the theorem, it is called the nonunits formal power series.

## A matrix representation of formal power series:

Let $P = a_0 + a_1x + a_2x^2 + \cdots$ ..., then the infinite triangular matrix is:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ 0 & 0 & 0 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

is called semicirculant matrix which is a special case of upper triangular matrix. The correspondence between the fps in P and the semicirculant matrix is $P \to A$.

- Some properties for formal power series: the sum of fps is associated i.e if $P \to A, Q \to B$, then $P + Q \to A + B$. Also, the same for the multiplication: if $P \to A, Q \to B$, then $PQ \to AB$. According to these properties, there is a map $\to$ of the fps onto semicirculants which is an isomorphism.
- Wronski formula theorem: if $P := a_0 + a_1x + a_2x^2 + \cdots$ ... is fps, and $a_0 \neq 0$, then the coefficients of the reciprocal series $P^{-1} =: b_0 + b_1x + b_2x^2 + \cdots$ ... .. are given by:

$$b_n = \frac{(-1)^n}{{a_0}^{n+1}} \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \vdots & a_{n-2} \\ \cdots & \cdots & & \cdots \\ 0 & 0 & a_0 & a_1 \end{vmatrix}, \text{ n=1,2,…..}$$

## Differentiation of formal power series:

Differentiation in the integral domain of fps acts on a sequence $P = \{a_0, a_1, a_2, \dots \dots\}$ such that if $P = a_0 + a_1x + a_2x^2 + \cdots$ ..., then the derivative of P is $P' := a_1 + 2a_2x + 3a_3x^2 + \cdots$ ... .

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- Theorems:
  1) If P is a fps over F such that $P^{'} = 0$, then P is a constant series.
  2) For arbitrary a,b in F, $E_a E_b = E_{a+b}$. (Addition theorem for the exponential series).

By these theorems we can establish Vandermonde's theorem:

$$\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}$$

**Formal hypergeometric series and finite hypergeometric sums:**

Let p,q are positive integers, and let $a_1, a_2, \ldots\ldots, a_p, b_1, b_2, \ldots\ldots, b_q$be elements in F. So, for all I and positive n's, $b_i + n \neq 0$, the fps:

$$\boldsymbol{F} \coloneqq \sum_{\boldsymbol{n=0}}^{\infty} \frac{(\boldsymbol{a_1})_{\boldsymbol{n}}(\boldsymbol{a_2})_{\boldsymbol{n}} \ldots\ldots (\boldsymbol{a_p})_{\boldsymbol{n}}}{(\boldsymbol{b_1})_{\boldsymbol{n}}(\boldsymbol{b_2})_{\boldsymbol{n}} \ldots\ldots (\boldsymbol{b_q})_{\boldsymbol{n}}\boldsymbol{n}!} \, \boldsymbol{x^n}$$

is called a generalized hypergeometric series, where $(a_i)$ is called numerator parameter and $(b_i)$ is called denominator parameter. For example, $E_a(x) = 1 + \frac{ax}{1!} + \frac{(ax)^2}{2!} + \cdots = {}_0F_0(ax)$.

- Theorem: let $a_1, a_2, \ldots\ldots, a_p, b_1, b_2, \ldots\ldots, b_q$be arbitrary elements of F such that no $b_i$ is zero or negative integers. Then the generalized hypergeometric series is the only solution of the formal differential equation $\theta(\theta + b_1 - 1) \ldots\ldots (\theta + b_q - 1)P = x(\theta + a_1)(\theta + a_p)P$ with zeroth coefficient 1.
- Example: the series ${}_1F_0(a;x)$ satisfies $\theta P = x(\theta + a)P$, i.e $(1-x)p^{'} = aP$. By this theorem, the series is the only formal solution with initial coefficient 1 of the equation $\theta(\theta + c - 1)P = x(\theta + a)(\theta + b)P$. Then

  ${}_1F_0(c-a-b;x)\ {}_2F_1(a,b;c;x) = {}_2F_1(c-a, cab; c; x)$ called Euler's first identity in the theory of hypergeometric series. On the left, the coefficient of $x^n$ is

  $\sum_{k=0}^{n} \frac{(a)_k (b)_k}{(c)_k k!} \frac{(c-a-b)_{n-k}}{(n-k)!} = \frac{(c-a-b)_n}{n!} \ {}_3F_2 \begin{bmatrix} a, b, -n; & 1 \\ c, a+b-c-n+1 & 0 \end{bmatrix}$.

But, the coefficient on the right is $\frac{(c-a)_n\,(c-b)_n}{(c)_n\,n!}$. After that, we will obtain the identity

${}_3F_2\left[\begin{matrix} a, b, -n; 1 \\ c, a+b-c-n+1 \end{matrix}\right] = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n}$ (Saalschutz Formula).

## The composition of a formal power series with a nonunit:

Let $P := a_0 + a_1x + a_2x^2 + \cdots \ldots$ and $Q \coloneqq b_0 + b_1x + b_2x^2 + \cdots \ldots$ are two formal power series. Now, substitute Q for x in P, we will get $Q^n$such that for all n=0,1,2,…… $Q^n = (b_0 + b_1x + b_2x^2 + \cdots \ldots)^n =: {b_0}^{(n)} + {b_1}^{(n)}x + {b_2}^{(n)}x^2 + \cdots \ldots$ . If we substitute these series for $x^n$ into P, we will get the coefficient of $x^n$ such that $c_n = a_0{b_n}^{(0)} + a_1{b_n}^{(1)} + a_2{b_n}^{(2)} + \cdots$

for n=0,1,2,…. .

$\Longrightarrow PoQ = c_0 + c_1x + c_2x^2 + \cdots \ldots$ is the composition of P with Q.

- Theorems:
  1) $c_n = \sum_{k+m\leq n} a_k b_m {p_n}^{(k,m)}$ intuitively means to calculate the product:
  $C = (a_0 + a_1Q + a_2Q^2 + \cdots \ldots)(b_0 + b_1R + b_2R^2 + \cdots \ldots)$ we must form all possible cross products, $C = a_0b_0 + a_1b_0Q + a_0b_1R + a_2b_0Q^2 + a_1b_1QR + \cdots \ldots$ expand each term separately, and collect coefficients of like powers. Since Q and R are nonunits, only the cross products $Q^kR^m$ where $k + m \leq n$ contribute to the coefficient of $x^n$.
  2) If A is a formal power series and Q a nonunits, $(AoQ)' = (A'oQ).Q'$ .
  3) IF $Q \coloneqq b_1x + b_2x^2 + \cdots \ldots$. And $B_a$ denotes the binomial series, then $B_aoQ = c_0 + c_1x + c_2x^2 + \cdots \ldots$. Where $c_0 = 1$, $c_n = \frac{1}{n}\sum_{k=1}^{n}[(a+1)k - n]\,c_{n-k}b_k$ , n=1,2,…. (J.C.P Miller formula).

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## The group of almost units under composition:

- Theorems:
  1) Under the operation of composition, the almost units in the integral domain of formal power series over a field F form a group.
  2) Let P and Q be nonunits and let $P \longrightarrow A, Q \longrightarrow B$. Then, $PoQ \rightarrow AB$. If $P =: a_1x + a_2x^2 + \cdots$ ... and $Q =: b_1x + b_2x^2 + \cdots$ ...., and $AB =: (c_{ij})$ then for $n \geq m$

$$c_{mn} = a_m{}^{(m)} b_n{}^{(m)} + a_{m+1}{}^{(m)} b_n{}^{(m+1)} + \cdots \ldots + a_n{}^{(m)} b_n{}^{(n)}$$

  3) If P is an almost unit, and $P \rightarrow A$, then $P^{[-1]} \rightarrow A^{-1}$.
  4) If P is an almost unit and $Q \coloneqq P^{[-1]}$, then $Q' = (P'oQ)^{-1}$.
  5) Let $E_a, B_a$ and L denote the exponential, binomial, and logarithmic series, respectively. Then $E_a oL = B_a$.

## Formal Laurent series; Residues:

- Definition of quotient field: it is defined as the set of equivalence classes of pairs (a,b)of elements of P, where $b \neq 0$ and two pairs (a,b) and (c,d) are equivalent if ad= bc.
- Example: the quotient field of the integral domain of integers is the field of rational numbers.
- Definition of formal Laurent series (fLs) over the field F: it is a sequence of elements of F whose indices the set of all integers under the condition of finite numbers of elements with negative indices are different from zero. The formal Laurent series $L \coloneqq \cdots..,0,0, a_k, a_{k+1}, \ldots$ where k is a negative integer and x ia an indeterminate, contains at most a finite number of power with negative exponents $L = a_k x^k + a_{k+1} x^{k+1} + \cdots..$ .
- The product of two formal Laurent series: if $L \coloneqq \sum a_1 x^1$ and $M \coloneqq \sum b_m x^m$ are two formal Laurent series. Then $LM \coloneqq \sum c_n x^n$, where $c_n \coloneqq \sum_{1+m=n} a_1 b_m$. For example, if k is any integer, then $\left(\sum_{n=k}^{\infty} x^n\right)^2 = \sum_{n=2k}^{\infty} (n - 2k + 1) x^n$.

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- Theorems:
  1) The formula Laurent series over F themselves from a field. If $L \coloneqq a_k x^k$ is a formal Laurent series, then $L' \coloneqq \sum (k+1) a_{k+1} x^k$ the coefficient $a_{k+1}$ is called the residue of L.
  2) A formula Laurent series is a derivative if and only if its residue is zero.

**The Lagrange- Burmann theorem:**

Let P and L denote the integral of fps and the field of fLs over F respectively. So, the reversion (inverse with respect to the composition) of the most unit $P \coloneqq a_1 x + a_2 x^2 + \cdots, \quad a_1 \neq 0$ in P can be found by finding the inverse of the matrix.

- Schur- Jabotinski theorem: let $P \coloneqq a_1 x + a_2 x^2 + \cdots \ldots$ be an almost unite in P $(a_1 \neq 0)$ and let $P^k =: \sum_{n=k}^{\infty} a_n{}^{(k)} x^n$ for $k = \pm 1, \pm 2, \ldots \ldots$ if $Q \coloneqq P^{[-1]}$, then for all positive integers m $Q^m = \sum_{n=m}^{\infty} b_n{}^{(m)} x^n$ where $b_n{}^{(m)} \coloneqq \frac{m}{n} \, a_{-m}{}^{(-n)}$ , $m \geq n.$
- Example: let $P \coloneqq XE_a = x + \frac{a}{1!} x^2 + \frac{a^2}{2!} x^3 + \cdots \ldots \ldots$ find the reversion of this series?
- Solution: since $p^k = X^k E_{ka}$ for all k $\Rightarrow a_n{}^{(k)} = \frac{(ka)^{n-k}}{(n-k)!}$ , $n \geq k$

$$\Rightarrow P^{[-1]} = \frac{1}{1} \, a_1{}^{(-1)} x + \frac{1}{2} a_1{}^{(-2)} x^2 + \frac{1}{3} \, a_1{}^{(-3)} x^3 + \cdots ..$$

$$= x + \frac{(-2a)^1}{2!} \, x^2 + \frac{(-3a)^2}{3!} \, x^3 + \cdots ....$$

Now, let P be an almost unit in P, and let $Q \coloneqq P^{[-1]}$ and $R \coloneqq c_0 + c_1 x + c_2 x^2 + \cdots ..$ (arbitrary)

$\Longrightarrow RoQ =: d_0 + d_1 x + d_2 x^2 + \cdots$ where $d_n = \sum_{k=1}^{n} c_k b_n{}^{(k)}$, n=1,2,3…….

By using Schur- Kabotinski theorem $\Rightarrow d_n = \frac{1}{n} \sum_{k=1}^{n} c_k k a_{-k}^{(-n)}$

$$\Rightarrow d_n = \frac{1}{n} \, res(R' P^{-n}), \quad n = 1,2, \ldots \ldots$$

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- Lagrange- Burmann expansion theorem: let P be an almost unit in P, $Q \coloneqq P^{[-1]}$ , and let $R \coloneqq c_0 + c_1 x + \cdots$.. be an arbitrary element of P. Then $RoQ = c_0 + \sum_{n=1}^{\infty} \frac{1}{n}\, res\,(R' P^{-n}) x^n$
- Proof: let $Q \coloneqq P^{[-1]}$ and P(0)=0, then Q(0)=0

Let $Q(x) = \sum_{n=1}^{\infty} q_n x^n$ and $x = Q\big(p(x)\big) = \sum_{n=1}^{\infty} q_n P(x)^n$

Therefore, 1=$\sum_{n=1}^{\infty} q_n P(x)^{n-1} P'(x)$

So, $\frac{1}{P^k(x)} = \sum_{n=1}^{\infty} q_n n P(x)^{n-k-1} P'(x)$

$Res\left(\frac{1}{P^k(x)}\right) = q_k$

Since $P(x)^{n-k-1} P'(x)$ has residue 0 at 0 unless n=k

Therefore, $Q(x) = \sum_{n=1}^{\infty} q_n x^n = \sum_{n}^{\infty} \frac{1}{n}\, Res\left(\frac{1}{P^n(x)}\right). X^n$

This proves the formula for $R(x) = x$

In general, suppose R: arbitrary, then let $RoQ$ have power series expansion: $\sum r_n x^n$

Then $RoQ(x) = R(x)$. So, $R(x) = \sum_{n=0}^{\infty} r_n P(x)^n$ and $R(0) = r_0$

Therefore, $R'(x) = \sum_{n=0}^{\infty} r_n n P(x)^{n-1} P'(x)$. So, $\frac{R'(x)}{P(x)^k} = \sum_{n=0}^{\infty} r_n P(x)^{n-k-1} P'(x)$

Therefore, $Res\left(\frac{R'(x)}{P(x)^k}\right) = r_k k, and\ k > 0$

So, $r_k = \frac{1}{k}\, Res\,(\frac{R'(x)}{P(x)^k})$ and $RoQ(x) = \sum_{n=1}^{\infty} \frac{1}{k}\, Res\left(\frac{R'(x)}{P(x)^k}\right) + R(0)$

- Corollary: for any $S \in P$ and any almost unit $P \in P$, IF $P^{[-1]} =: Q$ , there holds

$$(SoQ)Q' = \sum_{n=0}^{\infty} res(SP^{-n-1}) x^n$$

- Problem: let $F \coloneqq x - \frac{1}{3}\, x^3 + \frac{1}{5}\, x^5 - \cdots$ ..... and $P \coloneqq \frac{2x}{1-x^2}$ , $Q = P^{[-1]}$.

Prove the functional relation $FoQ = \frac{1}{2}\, F$

- SOLUTION: difficult by direct substitution. We prove $F = \frac{1}{2}\ FoP$.

Differentiate left hand side ⇒ $F'(x) = 1 - x^2 + x^4 - x^6 \ldots \ldots . = \frac{1}{1+x^2}$

Differentiate right hand side ⇒ $\frac{1}{2}F\big(P(x)\big)P'(x) = \frac{1}{2}\ \frac{1}{1+P(x)^2}\ P(x)$

$$= \frac{1}{2}\ \frac{1}{1 + \frac{4x^2}{(1-x^2)^2}} \left[\frac{(1-x^2)2 - 2x^{(-2x)}}{(1-x^2)^2}\right]$$

$$= \frac{1}{2}\ \frac{(1-x^2)^2}{(1-x^2)^2 + 4x^2}\ (\frac{2+2x^2}{(1-x^2)^2})$$

$$= \frac{1+x^2}{1-2x^2+x^4+4x^2}$$

$$= \frac{1+x^2}{(1+x^2)^2}$$

$$= \frac{1}{1+x^2}$$

So, $F(x) = \frac{1}{2}\ \frac{d}{dx}\ F\big(p(x)\big)$. Therefore, $F\big(p(x)\big) = \frac{1}{2}\ F\big(p(x)\big) + C$

$F(0) = 0, F\big(P(0)\big) = 0, so\ C = 0$

Therefore, $F = \frac{1}{2}\ FoP$

So, $FoQ = \frac{1}{2}\ F$

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# Result

This study indicated the ability of finding the solution of the following problem: let $F \coloneqq x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots \ldots\ldots$ and $P \coloneqq \frac{2x}{1-x^2}$ , $Q = P^{[-1]}$ and finding the functional relation to $FoQ = \frac{1}{2}F$ by proving that $F = \frac{1}{2}FoP$ instead of proving it directly.

Understanding the properties of formal power series, formal Laurent series, and Lagrange-Burmann theorem were useful to recognize the solution.

# Reference

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